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Regularly continuous and fully continuous mappings

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Abstract

The aim of this paper is to introduce the concepts of regularly continuous and fully continuous mappings as the mappings that have the preimages of regular open sets are regular open (respectively, open). We investigate the connections between these classes and several well-known others of 'generalized continuous mappings'. Several characterizations and decompositions of certain continuities are provided. In particular, a decomposition of complete continuity, which was first introduced by Arya and Gupta [1] , is also provided.

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1 Introduction

Let $(X,\mathcal{T})$ be a topological space. A subset $A \subseteq X$ is called *semi-open* if there exists an open set $O \in \mathcal{T}$ such that $O \subseteq A \subseteq \bar{O}$, where $\bar{O}$ denotes the closure of O in X. Clearly A is a semi-open set if and only if $A \subseteq \overline{\overset{o}{A}}$. A complement of a semi-open set is called *semi-closed*. A is called *preopen* if $A \subseteq \overset{o}{\overline{A}}$. An open set A is called *regular-open* if $A = \overset{o}{\overline{A}}$.

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Complements of regular-open sets are called *regular-closed.* Clearly A is regular-closed if and only if $A = \overline{\overset{o}{A}}$. A is called a *$\mathcal{B}$-set* if $A = O \cap B$, where O is open and B is semi-closed. A is called a *t-set* if $\overset{o}{\overline{A}} = \overset{o}{A}$. For the following definition and for more on the preceding notions, see for example [9, 10, 11, 13, 14, 15]

Definition 1 *Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}')$ be topological spaces. A mapping $f : X \to Y$ is* t-continuous *(respectively,* completely continuous, precontinuous*) if the preimage of every open set in Y is a t-set (respectively, regularly open, preopen) in X.*

Next, we recall the following fundamental result.

Lemma 1 [15] *A subset A of a topological space X is regular open if and only if A is preopen and a t-set.*

The notion of continuity is one of the most used concepts in mathematics. Many decompositions of continuity were introduced, see for example [8, 9, 10, 11, 13, 14, 15]. An obvious problem is to look for new decompositions of weaker forms of continuity in order to establish new decompositions of continuity, for example a decomposition of complete continuity, which was first introduced by Arya and Gupta [1] , is provided in this paper. In [6], the concept of ω-closed subsets was explored where a subset A of a space $(X, \mathfrak{T})$ is ω-closed if it contains all of its condensation points. In [7], several characterizations of ω-closed subsets were provided. In [1], the concept of $\epsilon - open$ was introduced and explored and in [2], sevral decompositions of continuity via this notion were provided. In [3], the concept of $\rho - open$ was introduced and explored and sevral decompositions of continuity via this notion were provided. Also in [5], a similar work was done for what is called $\xi - open$ sets.

In section 2, regular continuity and fully continuity concepts are introduced. We show that the concept of fully continuity is weaker than both continuity and regular continuity, while continuity and regular continuity are independent. In section 3, the notions of continuity and t-continuity are shown to be independent. Moreover, restrictions and compositions of t-continuous mappings are studied. Sections 4 and 5 are devoted to compositions and restrictions of regularly continuous and fully continuous mappings, respectively. In section 6, complete continuity is shown to be a stronger notion than both continuity and regular continuity and hence, stronger than fully continuity. In addition, a decomposition of complete continuity is provided. While in section 7, several characterizations and a decomposition of certain types of continuities are given.

2 Regular continuity and fully continuity

In this section, the notions of *regularly continuous and fully continuous mappings* are introduced. The relations between these concepts and continuity, t-continuity and complete continuity are discussed.

Definition 2 *Let* $(X, \mathfrak{T})$ *and* $(Y, \mathfrak{U})$ *be topological spaces. Then a mapping* $f : X \to Y$ *is* regularly *(respectively,* fully*) continuous if the preimage of every regular open set is regular open (respectively, open).*

t-continuity was first introduced by Tong [15] and precontinuity was first introduced by Mashhour, Abd El-Monsef and El-Deeb [12], while *regular continuity* and *fully continuity* seem to be new notions.

As every regular open set is open, every continuous mapping and every regularly continuous mapping is fully continuous. We now provide examples to distinguish these concepts.

Example 1 *Let* $X = \{a, b, c, d\}$ *and* $\mathfrak{T} = \{\varnothing, X, \{d\}, \{a, b\}, \{a, b, d\}\}$*. Then the mapping* $f : X \to X$ *defined by* $f(a) = f(b) = f(d) = d$ *and* $f(c) = c$ *is continuous and hence, fully continuous.* f *is not regularly continuous as* $\{d\}$ *is regular open, but* $\{a, b, d\} = f^{-1}(\{d\})$ *is not.*

Example 2 *Let* $X = \{a, b, c, d\}$ *and* $\mathfrak{T} = \{\varnothing, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}$ $, \{a, c, d\}\}$*. Then the mapping* $f : X \to X$ *defined by* $f(a) = b$*,* $f(b) = a$*,* $f(c) = c$ *and* $f(d) = d$ *is regularly continuous, but not continuous.*

A fully continuous mapping needs not be continuous as otherwise, every regularly continuous mapping is continuous. The next result is fundamental and will be used several times throughout this paper.

Theorem 1 *A subset A of a topological space X is regular open if and only if A is open and a t-set.*

Proof. If A is a regular open subset of X, then A is open. Also by Proposition 2 in Tong [15], A is a t-set. Conversely, an open t-set is clearly a regular open set. □

Theorem 2 *If $f : X \to Y$ is continuous and t-continuous, then f is regularly continuous.*

Proof. Let A be a regular open subset of Y. Then A is open and hence $f^{-1}(A)$ is both open and a t-set. By Theorem 1, $f^{-1}(A)$ is regular open and thus, f is regularly continuous. □

Note that the converse of the preceding theorem need not be true, see Example 2.

Definition 3 *A mapping $f : X \to Y$ is* t-open (*respectively,* almost preopen) *if the image of every open set in X is a t-set (respectively, regular open) in Y.*

Next, a decomposition of almost preopen mappings is given.

Theorem 3 *A mapping $f : X \to Y$ is open and t-open if and only if f is almost preopen.*

Proof. Let A be an open set in X. Then as f is open and t-open, $f(A)$ is open and a t-set. Hence by Theorem 1, $f(A)$ is regular open and so f is almost preopen. Conversely, if f is almost preopen and A is an open set in X, then $f(A)$ is regular open. By Theorem 1, $f(A)$ is open and a t-set. Therefore, f is open and t-open. □

3 t-continuity

Continuity and t-continuity are independent as shown by the following two examples.

Example 3 *Let* $X = \{a, b, c\}$ *and* $\mathfrak{T} = \{\varnothing, X, \{a\}, \{a, b\}\}$. *Then the mapping* $f : X \to X$ *defined by* $f(a) = f(b) = b$ *and* $f(c) = a$ *is t-continuous.* f *is not continuous as* $\{a\}$ *is open, but* $\{c\} = f^{-1}(\{a\})$ *is not open, but the preimage of any open set is a t-set, thus* f *is t-continuous.*

Example 4 *Let* $X = \{a, b, c\}$ *and* $\mathfrak{T} = \{\varnothing, X, \{a\}, \{a, b\}\}$. *Then the identity mapping* $f : X \to X$ *is continuous and not t-continuous as* $\{a\}$ *is open but* $\{a\}$ *is not a t-set.*

Theorem 4 *If* $f : X \to Y$ *is t-continuous and precontinuous, then* f *is continuous.*

Proof. Let A be an open set in Y. As f is t-continuous, $f^{-1}(A)$ is a t-set and hence by Proposition 5 in Tong [15], $f^{-1}(A)$ is a $\mathcal{B}$-set. Also as f is precontinuous, $f^{-1}(A)$ is preopen. Thus by Proposition 9 in Tong [15], $f^{-1}(A)$ is open and thus f is continuous. □

Restrictions of t-continuous mappings need not be t-continuous as shown by the following example.

Example 5 *Consider the topological space* $(X, \mathfrak{T})$ *as in Example 1 and the mapping* $f : X \to X$ *defined by* $f(a) = f(b) = a$ *and* $f(c) = f(d) = c$. *Clearly,* f *is t-continuous. Set* $A = \{b, c\}$. *Then as* $\{a, b\}$ *is open in* X *and as* $\{b\} = (f|_A)^{-1}(\{a, b\})$ *is not a t-set in* $(A, \mathfrak{T}_A)$, $f|_A$ *is not t-continuous.*

Compositions of t-continuous mappings need not be t-continuous as shown by the following example.

Example 6 *Consider the topological space* $(X, \mathfrak{T})$ *and the t-continuous mapping* f *as in Example 3. Define the mapping* $g : X \to X$ *by* $g(a) = c$, $g(b) = b$ *and* $g(c) = a$. *Then* g *is t-continuous.* $g \circ f$ *is not t-continuous as* $\{a, b\}$ *is open, but* $\{a, b\} = (g \circ f)^{-1}(\{a, b\})$ *is not a t-set.*

Next, we provide three new results in which we give sufficient conditions on mappings that when composed with t-continuous mappings, the results will be t-continuous mappings.

Theorem 5 *If $f : X \to Y$ is t-continuous and $g : Y \to Z$ is continuous, then $g \circ f$ is t-continuous.*

Proof. Let A be an open subset of Z. Since g is continuous, $g^{-1}(A)$ is open in Y and then as f is t-continuous, $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$ is a t-set. Therefore, $g \circ f$ is t-continuous. □

Corollary 1 *If $f : X \to Y$ is t-continuous and $g : Y \to Z$ is complete continuous, then $g \circ f$ is t-continuous.*

Proof. Follows from the fact that every regular open set is open. □

Corollary 2 *If $f : X \to Y$ is t-continuous and $g : Y \to Z$ is t-continuous and precontinuous, then $g \circ f$ is t-continuous.*

Proof. Follows from the fact that every t-set is a $\mathcal{B}$-set and then as it is preopen, by Proposition 9 in Tong [15], it is open. □

4 More properties of regular continuity

In this section, we study compositions and restrictions of regularly continuous mappings. In particular, the restriction of a regularly continuous mapping needs not be regularly continuous as shown by the following example.

Example 7 *Consider the topological space $(X, \mathfrak{T})$ from Example 1 and the mapping $f : X \to X$ defined by $f(a) = f(b) = a$, $f(c) = c$ and $f(d) = d$. Then f is regularly continuous. Set $A = \{b, c\}$. Then $\{a, b\}$ is regular open in X, but $\{b\} = (f|_A)^{-1}(\{a, b\})$ is not regular open in $(A, \mathfrak{T}_A)$. Hence, $f|_A$ is not regularly continuous.*

The proof of the following result is obvious.

Theorem 6 *A composition of two regularly continuous mappings is regularly continuous.*

Theorem 7 *If $f : X \to Y$ is regularly continuous and $g : Y \to Z$ is completely continuous, then $g \circ f$ is regularly continuous.*

Proof. Let B be a regular open set in Z. Then as B is open in Z and as $g : Y \to Z$ is completely continuous, $g^{-1}(B)$ is a regular open set in Y. Hence as f is regularly continuous, $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$ is a regular open set in X. Therefore, $g \circ f$ is regularly continuous. □

5 More properties of fully continuity

In this section, we study compositions and restrictions of fully continuous mappings. In particular, the restriction of a fully continuous mapping needs not be fully continuous as shown by the following example.

Example 8 *Consider the topological space from Example 1 and the mapping $f : X \to X$ defined by $f(a) = f(b) = a$ and $f(c) = f(d) = c$. Clearly, f is fully continuous. Set $A = \{b, c\}$. Then as $\{a, b\}$ is a regular open set in X and as $\{b\} = (f|_A)^{-1}(\{a, b\})$ is not open in $(A, \mathfrak{T}_A)$, $f|_A$ is not fully continuous.*

Compositions of fully continuous mappings need not be fully continuous as shown by the following example.

Example 9 *Let $X = Y = Z = \{a, b, c, d\}$, $\mathfrak{T}_X = \{\varnothing, X\}$, $\mathfrak{T}_Y = \{\varnothing, Y, \{a, b, c\}\}$ and $\mathfrak{T}_Z = \{\varnothing, Z, \{d\}, \{a, b\}, \{a, b, d\}\}$. Then the mappings $f : X \to Y$ and $g : Y \to Z$ defined by $f(a) = f(b) = f(c) = d$, $f(d) = c$, $g(a) = g(b) = g(c) = d$ and $g(d) = c$ are fully continuous but $g \circ f$ is not fully continuous as $\{d\}$ is a regular open set in Z and $\{d\} = (g \circ f)^{-1}(\{d\})$ is not open in X.*

Example 3 shows that t-continuity does not imply regular continuity. Example 4 shows that neither regular continuity nor fully continuity imply t-continuity. Thus on one hand, t-continuity and regular continuity are independent but on the other hand, t-continuity and fully continuity are independent.

6 Complete continuity

As every regular open set is open, complete continuity implies continuity. We now provide an example to distinguish these concepts.

Example 10 [8] *Let* $X = \{a, b, c, d\}$, $\mathfrak{T}_X = \{\varnothing, X, \{a, b\}\}, Y = \{p, q\}$, $\mathfrak{T}_Y = \{\varnothing, Y, \{p\}\}$. *Let* $f : X \to Y$ *be defined by* $f(a) = f(b) = p$ *and* $f(c) = f(d) = q$*. Then* f *is continuous but not completely continuous.*

Theorem 8 *If* $f : X \to Y$ *is completely continuous, then* f *is regularly continuous.*

Proof. Let B be a regular open set in Y. Then B is open and as f is completely continuous, $f^{-1}(B)$ is regular open set. Therefore, f is regularly continuous. ∎

Corollary 3 *Every completely continuous mapping is fully continuous.*

As an open set needs not be regular open, a regularly (fully) continuous mapping needs not be completely continuous. For a complete discussion of compositions and restrictions of complete continuous mappings, see Arya and Gupta [8].

Theorem 3 seems to be a new result and provides the following new decomposition of complete continuity.

Theorem 9 *A mapping* $f : X \to Y$ *is completely continuous if and only if* f *is continuous and t-continuous.*

7 A decomposition of continuity

We begin this section with the following basic definition.

Definition 4 *A space X is a* T-space *if every open set in X is a t-set.*

That is, T-spaces are precisely those spaces in which every open set

is closed or equivalently those spaces in which every subset is preopen. Thus, every discrete space is a T-space. If X is a space in which every open set is regular open, then by Proposition 2 in Tong [15], X is a T-space. In all of what follows, X stands for a topological space while X^* stands for a T-space.

Next, three new characterizations of special continuities are given.

Theorem 10 *A mapping $f : X \to Y^*$ is continuous if and only if f is fully continuous.*

Proof. Continuity implies fully continuity is trivial. Let B be an open subset of Y^*. Then B is a t-set and by Theorem 1, B is regular open. Thus as f is fully continuous, $f^{-1}(B)$ is open. Therefore, f is continuous. □

Theorem 11 *A mapping $f : X^* \to Y$ is continuous if and only if f is completely continuous.*

Proof. Completely continuity implies continuity is trivial. Let B be an open subset of Y. Then as f is continuous, $f^{-1}(B)$ is open in X^* and hence a t-set. Thus by Theorem 1, $f^{-1}(B)$ is regular open. Therefore, f is completely continuous. □

Theorem 12 *A mapping $f : X^* \to Y^*$ is continuous if and only if f is regularly continuous.*

Proof. Let B be a regular open subset of Y^*. Then B is open and as f is continuous, $f^{-1}(B)$ is open in X^* and hence a t-set. Thus by Theorem 1, $f^{-1}(B)$ is regular open. Therefore, f is regularly continuous. Conversely, if f is regularly continuous and B is an open subset of Y^*, then B is a t-set and by Theorem 1, B is regular open. Thus as f is regularly continuous, $f^{-1}(B)$ is regular open and hence open. Therefore, f is continuous. □

We end this section with the following decomposition of continuity.

Theorem 13 *A mapping $f : X^* \to Y$ is continuous if and only if f is t-continuous and precontinuous.*

Proof. Let B be an open subset of Y. Then as f is continuous, $f^{-1}(B)$ is open in X^* and hence a t-set. Thus, f is t-continuous. Moreover, $f^{-1}(B) = (f^{-1}(B))^o = \overset{o}{\overline{(f^{-1}(B))}}$ and hence $f^{-1}(B)$ is preopen. Therefore, f is precontinuous.

Conversely, if f is t-continuous and precontinuous, then by Theorem 4, f is continuous. □

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